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Logarithmic foliations on compact algebraic surfaces

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Abstract. Let $\mathcal{F}$ be a holomorphic singular foliation with an invariant compact curve S on a compact algebraic surface X. In this work we prove that $\mathcal{F}$ is logarithmic under some generic conditions in the singularities of $\mathcal{F}$ in S.

Keywords: Holomorphic foliation, Indices of singularities, Invariant compact curves, Hodge index theorem.

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1 Introduction

Let X be a smooth compact algebraic surface and let $\mathcal{F}$ be a holomorphic foliation with isolated singularities on X. Denote by $\mathrm{Sing}(\mathcal{F})$ the set of singular points of $\mathcal{F}$. We say that a compact curve S is invariant by $\mathcal{F}$ provided $S \setminus \mathrm{Sing}(\mathcal{F})$ is a leaf of the foliation $\mathcal{F}/(X \setminus \mathrm{Sing}(\mathcal{F}))$.

The purpose of this work is to give some characterizations of $\mathcal{F}$ in terms of its singularities in S.

Recall that $\mathcal{F}$ is said logarithmic if there exist $\{\lambda_i\}_{1\leq i\leq n} \subseteq \mathbb{C}^*$ and $\{S_i\}_{1\leq i\leq n}$ irreducible curves such that $\mathcal{F}$ is locally given by the holomorphic 1-form

$$\omega = f_1 \dots f_n \sum_{1\leq i\leq n} \lambda_i \frac{df_i}{f_i} \quad \text{where the } f_i \text{ define locally } S_i.$$

In [CL] Theorem 1 is given a characterization of a logarithmic foliation in terms of its singularities in an invariant compact curve. More precisely it is proved that, if $\mathcal{F}$ is a holomorphic foliation on $\mathbb{CP}^2$ of degree m and S is an

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invariant compact curve of degree n holds the relation $n \leq m+2$ provided that S has only nodal type singularities (that is, singularities of normal crossing type). Moreover if the equality holds, then $\mathcal{F}$ is logarithmic.

In this work we obtain another characterization of a logarithmic foliation as in [CL], but we allow that S has others type of singularities. However we must restrict ourselves to the case that the singularities of $\mathcal{F}$ in S are non-dicritical generalized curves, that is, in its resolution by blow ups does not appear any saddle node ([CLS]).

More precisely, if $Pic(X) := H^1(X, \mathcal{O}_X^*)$ is the Picard group of X and S^2 is the self-intersection number of S, then we prove

Theorem A. *Let $\mathcal{F}$ be a holomorphic foliation on a compact algebraic surface X and let S be an invariant compact curve by $\mathcal{F}$. Assume that one of the following conditions hold:*

i) Pic(X) is isomorphic to $\mathbb{Z}$ or

ii) Pic(X) is torsion free, $H^1(X, \mathbb{C}) = 0$, $S^2 > 0$ and

$$\sum_{p \in \mathrm{Sing}(\mathcal{F})-S} BB_p(\mathcal{F}) \geq 0.$$

Then, if every local separatrix of $\mathcal{F}$ through any $p \in \mathrm{Sing}(\mathcal{F}) \cap S$ is a local branch of S and if every singularity of $\mathcal{F}$ in S is a generalized curve, $\mathcal{F}$ is logarithmic.

Here, the symbol $BB_p(\mathcal{F})$ stands for the Baum-Bott index associated to the Chern number c_1^2 of the normal sheaf of the foliation ([BB]).

We observe that the condition

$$\sum_{p \in \mathrm{Sing}(\mathcal{F})-S} BB_p(\mathcal{F}) \geq 0$$

holds if each singularity of $\mathcal{F}$ in $X \setminus S$ is linearly of Morse type (i.e. $\mathcal{F}$ is locally given by the holomorphic 1-form $d(xy) + h.o.t.$). This condition also holds when $\mathcal{F}$ has local holomorphic first integral around each point of X which is not in S.

Now let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{CP}^2$ of degree m, then

$$\sum_{p \in \mathrm{Sing}(\mathcal{F})} BB_p(\mathcal{F}) = (m+2)^2.$$

Therefore, we have that $\sum_{p\in \mathrm{Sing}(\mathcal{F})} BB_p(\mathcal{F}) \le S^2$ is equivalent to $m+2 \le n$, where n is the degree of S. Consequently we have the following extension of the second part of theorem 1 in [CL] to compact complex surfaces.

Proposition 3.1. *Let $\mathcal{F}$ be a holomorphic foliation on a compact algebraic surface X with $H^1(X,\mathbb{C}) = 0$ and $Pic(X) = \mathbb{Z}$. Let S be an invariant compact curve with only nodal type singularities. If*

$$\sum_{p\in \mathrm{Sing}(\mathcal{F})} BB_p(\mathcal{F}) \le S^2,$$

then $\mathcal{F}$ is logarithmic.

This work is divided as follows: in section 2 we briefly review the Baum-Bott index associated to a singularity and obtain an obstruction to the existence of an invariant compact curve in terms of this index and the self-intersection number of the curve. In section 3 we give the characterization of a logarithmic foliation in terms of its singularities in an invariant compact curve.

2 Preliminaries

A holomorphic singular foliation $\mathcal{F}$ with isolated singularities on X can be given by a family of holomorphic 1-forms $\{\omega_i\}_{i\in I}$ defined on an open covering of X, $\mathcal{U} = \{\mathcal{U}_i\}_{i\in I}$, satisfying $w_i = g_{ij} w_j$ in $\mathcal{U}_i \cap \mathcal{U}_j$, $g_{ij} \in \mathcal{O}^*(\mathcal{U}_i \cap \mathcal{U}_j)$. We will denote by $N_{\mathcal{F}}$ the holomorphic line bundle on X obtained from the cocycle $\{g_{ij}\}$. This line bundle extends to X the line bundle normal to $\mathcal{F}$ on $X \setminus \mathrm{Sing}(\mathcal{F})$.

Let $\pi\colon \tilde{X} \to X$ be the blowing-up at $p \in X$ and $D = \pi^{-1}(p) \subseteq \tilde{X}$ the exceptional line. The foliation $\pi^*(\mathcal{F}/(X - \{p\}))$ of $\tilde{X} - D$ has an unique extension to a holomorphic foliation with isolated zeros on $\tilde{X}$ which we denote by $\tilde{\mathcal{F}}$ and call the strict transformed foliation of $\mathcal{F}$ by π. $\tilde{\mathcal{F}}$ is constructed in the following way, if $w = 0$ represents locally $\mathcal{F}$, w with isolated zero at p, then $\tilde{w} = \pi^*(w)$ has along D zeros of order $m_p \ge 1$. Therefore, if locally D is given by the equation $\{f = 0\}$, then the holomorphic 1-form $\Omega = \tilde{\omega}/f^{m_p}$ has isolated zeros and we define $\tilde{\mathcal{F}}$ by the holomorphic 1-form Ω.

We will use an additive notation for the Picard group of X, hence if $\mathcal{L} \in Pic(X)$ is a holomorphic line bundle and $m \in \mathbb{Z}$ the symbol, $m\mathcal{L}$ will denote the holomorphic line bundle $\mathcal{L}^{\otimes m}$. Now, it is not difficult to see that the effect of the blowing-up π on the line bundle above introduced is:

$$N^*_{\tilde{\mathcal{F}}} = \pi^*(N^*_{\mathcal{F}}) \otimes m_p \mathcal{O}_{\tilde{X}}(D), \tag{2.1}$$

where $N^*_{\mathcal{F}}$ is the dual line bundle of $N_{\mathcal{F}}$, and $\mathcal{O}_{\tilde{X}}(D)$ is the holomorphic line bundle defined on $\tilde{X}$ by the divisor D.

In this work a singularity p of $\mathcal{F}$ is called *dicritical* if the exceptional line introduced by one blowing-up is not invariant by $\tilde{\mathcal{F}}$, otherwise p is called *non dicritical*. We recall that $m_p = \nu_p(\mathcal{F})$ if p is *non-dicritical* and $m_p = \nu_p(\mathcal{F})+1$, if p is *dicritical*, where $\nu_p(\mathcal{F})$ is the order of the first non-zero jet of w at p which is called the *algebraic multiplicity* of $\mathcal{F}$ at p.

Associated with a singularity p of $\mathcal{F}$ we have the Baum-Bott index which permits to measure how the line bundle $N_{\mathcal{F}}$ is related with the singularities of the foliation.

Let us denote $BB_p(\mathcal{F})$ the Baum-Bott index at p, that is, if $p \in X$ is an isolated singularity of $\mathcal{F}$ and if $\mathcal{F}$ is given locally by the holomorphic 1-form

$$\omega = P(x,y)dy - Q(x,y)dx \quad \text{with} \quad gcd(P,Q) = 1 \quad \text{and} \quad p = (0,0)$$

then,

$$BB_0(\mathcal{F}) := \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \frac{(TrJ(x,y))^2}{P(x,y)Q(x,y)} dxdy,$$

where $J(x,y)$ is the Jacobian matrix of (P,Q), $\Gamma = \{(x,y) \mid |P(x,y)| = |Q(x,y)| = \epsilon\}$ for sufficiently small number $\epsilon > 0$ and Γ is oriented in such a way that the form $d(argP) \wedge d(argQ)$ is positive.

We note that $BB_p(\mathcal{F}) = 0$ holds if $\mathcal{F}$ has holomorphic first integral in a neighbourhood of p. We also have $BB_p(\mathcal{F}) = 0$ when $\mathcal{F}$ is linearly of Morse type in p (that is in a neighbourhood of p $\mathcal{F}$ is given by the holomorphic 1-form $d(xy) + h.o.t.$).

Now, set $BB(\mathcal{F}) := \sum_{p \in \mathrm{Sing}(\mathcal{F})} BB_p(\mathcal{F})$, then the Baum-Bott formula (see [BB]) asserts:

$$BB(\mathcal{F}) = c_1^2(N_{\mathcal{F}}), \tag{2.2}$$

where $c_1(.)$ denotes the first Chern class.

The next proposition gives an obstruction to the existence of compact invariant curves for holomorphic foliations. First we recall the Gomez Mont-Seade-Verjovsky index. Let S be an invariant compact curve by $\mathcal{F}$. Given a singularity p of $\mathcal{F}$ in S, in [GSV] it is introduced an index which is a kind of generalization of the usual Poincaré index associated to a vector field. This index enable us to relate how the restriction of the line bundle $N_{\mathcal{F}}$ to an invariant compact curve differs from the self-intersection number of the curve.

In fact, if $GSV_p(\mathcal{F}, S) \in \mathbb{Z}$ is the above index in $p \in \mathrm{Sing}(\mathcal{F})$, it was proved in [Br1], Lemma 3, that

$$GSV(\mathcal{F}, S) := \sum_{p \in S \cap \mathrm{Sing}(\mathcal{F})} GSV_p(\mathcal{F}, S) = c_1(N_{\mathcal{F}})S - S^2, \tag{2.3}$$

where S^2 denotes the self-intersection number of S.

Proposition 2.1. *Let $\mathcal{F}$ be a holomorphic foliation on a compact algebraic surface X and let S be an invariant compact curve with non-negative self-intersection number. If $GSV(\mathcal{F}, S) = 0$, then*

$$BB(\mathcal{F}) \leq S^2$$

Proof. By Baum-Bott formula we know that $BB(\mathcal{F}) = c_1^2(N_{\mathcal{F}})$. On the other hand by (2.3) $c_1(N_{\mathcal{F}})S = S^2$, which implies $c_1(N_{\mathcal{F}} \otimes \mathcal{O}_X(-S))S = 0$. Now, by the Hodge index theorem (see [BPV] pag. 120) we conclude that $c_1^2(N_{\mathcal{F}} \otimes \mathcal{O}_X(-S)) \leq 0$. Indeed if $c_1^2(N_{\mathcal{F}} \otimes \mathcal{O}_X(-S)) > 0$ then S would be homologous to zero in $H^2(X, \mathbb{Q})$. Hence $SL = 0$ for every divisor L on X. We will prove that this is impossible. Embed X in $\mathbb{CP}^N$, by Bertini's Theorem there exists a hyperplane H in $\mathbb{CP}^N$ transversal to both X and S and such that $H \cap S \neq \emptyset$. Let $L = X \cap H$, then $L \cap S$ is a discrete set of points, $\{x_i\}_{1 \leq i \leq n}$ and therefore

$$SL = \sum_{1 \leq i \leq n} [x_i] = n[p],$$

where $[x_i]$ and $[p]$ are the homological class of x_i and a point respectively in $H_0(X, \mathbb{Z})$. Then, $SL \neq 0$ contradiction. Thus $c_1^2(N_{\mathcal{F}} \otimes \mathcal{O}_X(-S)) \leq 0$, that is,

$$c_1^2(N_{\mathcal{F}}) + S^2 - 2c_1(N_{\mathcal{F}})S \leq 0$$

which implies, by (2.2) and (2.3) that $BB(\mathcal{F}) \leq S^2$. □

We say that a singularity of a holomorphic foliation is of radial type if the holomorphic foliation can be given locally by the holomorphic 1-form $\omega = xdy - ydx$. We say that a holomorphic foliation on $\mathbb{CP}^2$ is the radial foliation if all its leaves are projective lines that meet at one point. Let m be the degree of a foliation on $\mathbb{CP}^2$ as was introduced in [L], then a holomorphic foliation on $\mathbb{CP}^2$ is the radial foliation iff m is equal to zero.

As corollary of the above proposition we have

Corollary 2.1. *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{CP}^2$ of degree m and let S be an invariant compact curve. If each singularity of $\mathcal{F}$ in S is of radial type and if $\#\mathrm{Sing}(\mathcal{F}) \cap S \leq m + 1$, then $\mathcal{F}$ is the radial foliation.*

Proof. Let X be the compact complex surface obtained by one blow up at each singularity of $\mathcal{F}$ in S and denote $\pi : X \to \mathbb{CP}^2$ that sequence of blow ups. Let $\tilde{\mathcal{F}}$

be the strict transformed foliation of $\mathcal{F}$ by π and let $\tilde{S}$ be the strict transform of S by π. Then, $\tilde{\mathcal{F}}$ leaves invariant $\tilde{S}$ and $\mathrm{Sing}(\tilde{\mathcal{F}}) \cap \tilde{S} = \emptyset$. Denote $\{D_p\}_{p \in \mathrm{Sing}(\mathcal{F}) \cap S}$ the set of exceptional lines introduced by π. We get by (2.1) and Baum-Bott formula (2.2):

$$
\begin{aligned}
BB(\tilde{\mathcal{F}}) &= c_1^2(N^*_{\tilde{\mathcal{F}}}) \\
&= (\pi^* c_1(N^*_{\mathcal{F}}) + \sum_{p \in \mathrm{Sing}(\mathcal{F}) \cap S} 2D_p)^2 \\
&= c_1^2(N_{\mathcal{F}}) - 4\#(\mathrm{Sing}(\mathcal{F}) \cap S).
\end{aligned}
$$

On the other hand, from the index theorem of Camacho-Sad ([CS]), we have $\tilde{S}^2 = 0$. Now, by proposition 2.1 we conclude that $c_1^2(N_{\mathcal{F}}) \leq 4\#(\mathrm{Sing}(\mathcal{F}) \cap S)$.

But $(2+m)^2 = c_1^2(N_{\mathcal{F}})$ (for an elementary proof of this fact see [LS] pag. 94). Therefore $(m+2)^2 \leq 4(m+1)$. That is, $m = 0$ and consequently $\mathcal{F}$ is the radial foliation. □

Remark 2.1. The condition $\#\mathrm{Sing}(\mathcal{F}) \cap S \leq m+1$ in the above corollary is necessary as it is showed in the next example: Let $\mathcal{F}$ be the pencil generated by two quadrics which intersect transversally. Then, if S is one of the quadrics, we have that all singularities of $\mathcal{F}$ in S are radial. But here $m = 2$ and $\#\mathrm{Sing}(\mathcal{F}) \cap S = 4$.

3 Proof of the results

First recall the following result of Deligne.

Let X be a compact smooth algebraic variety of dimension n. Let $D = \sum_{1 \leq i \leq k} S_i$ be a reduced normal crossing divisor on X. That is, $\{S_i\}_{1 \leq i \leq k}$ are smooth codimension-one analytic sets on X and they intersect everywhere transversally.

A 1-form with at most a logarithmic pole along D is, by definition, a linear combination of $\frac{dz_1}{z_1}, \ldots, \frac{dz_r}{z_r}, dz_{r+1}, \ldots, dz_n$ with coefficients holomorphic functions. The set of all local 1-forms as above define a holomorphic sheaf of rank n which we call $\Omega^1_X(log D)$. If we set $\Omega^p_X(log D) := \bigwedge^p \Omega^1_X(log D)$ we have a complex with the usual differentiation. A result of Deligne asserts that, if $\Omega \in H^0(X, \Omega^p_X(log D))$ (that is, Ω is a global section of the sheaf $\Omega^p_X(log D)$), then Ω is closed (see [D] pag. 39).

The sheaf $\Omega^p_X(log D)$ can be alternatively defined in the following way. Let f be any local defining function for D. Then, $\Omega^p_X(log D)$ is the set of meromorphic p-forms Ω such that $f\Omega$ and $f d\Omega$ are holomorphic forms in X (see [D] page 31).

Lemma 3.1. *Let X be a compact algebraic surface with $H^1(X, \mathbb{C}) = 0$ and Picard group torsion free. Let $\mathcal{F}$ be a holomorphic foliation on X and let S be an invariant compact curve with only nodal type singularities. If the holomorphic line bundle $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S)$ is holomorphically trivial, then $\mathcal{F}$ is logarithmic.*

Proof. Let $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha\in\Lambda}$ be an open covering of X such that $S \cap \mathcal{U}_\alpha = \{f_\alpha = 0\}$ with f_α reduced and $\mathcal{F}$ given in $\mathcal{U}_\alpha$ by the holomorphic 1-form w_α.

We know that $g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ with $w_\alpha = g_{\alpha\beta} w_\beta$ are the transition functions of $N_{\mathcal{F}}$ and that $f_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ with $f_\alpha = f_{\alpha\beta} f_\beta$ are the transition functions of $\mathcal{O}_X(S)$. Then, since $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S)$ is a holomorphically trivial line bundle there exist $\rho_\alpha \in \mathcal{O}^*(\mathcal{U}_\alpha)$ such that $g^{-1}_{\alpha\beta} f_{\alpha\beta} = \rho_\alpha \rho^{-1}_\beta$ for every pair $\{\alpha, \beta\}$. Then, there is a global meromorphic 1-form Ω defined in each $\mathcal{U}_\alpha$ by

$$\Omega|_{\mathcal{U}_\alpha} = \frac{\rho_\alpha \omega_\alpha}{f_\alpha}.$$

This global meromorphic 1-form has its divisor of poles along S, moreover $f_\alpha \Omega$ is a holomorphic 1-form for every α. We also have that $f_\alpha d\Omega$ is a holomorphic 1-form for every α. Indeed since S is an invariant compact curve by $\mathcal{F}$ and f_α is reduced we get that $w_\alpha \wedge df_\alpha = f_\alpha \mu_\alpha$ for some holomorphic 2-form μ_α. Then,

$$f_\alpha d\Omega = \rho_\alpha d w_\alpha + d\rho_\alpha \wedge w_\alpha - \rho_\alpha \mu_\alpha.$$

Therefore $f_\alpha d\Omega$ is holomorphic. Since S has only nodal type singularities and Ω and $d\Omega$ have simple poles along S we have, by the aforementioned result of Deligne, that Ω is closed. Let $S_1, S_2, \ldots, S_n$ be the decomposition of the polar divisor of Ω in its irreducible components. For each S_j take a section $f_j \in \mathcal{O}_X(S_j)$ and set $\lambda_j := \frac{1}{2\pi i}\int_{\gamma_j} \Omega$, where γ_j is a closed curved in $X - S_j$ oriented in such a way that

$$\frac{1}{2\pi i}\int_{\gamma_j} \frac{df_j}{f_j} = 1.$$

Now, by the residue theorem, we have

$$\sum_{1\le i\le n} \lambda_i c_1(S_i) = 0. \tag{3.1}$$

On the other hand, since the Picard group of X is torsion free we know that there exist holomorphic line bundles $\mathcal{L}_i$, $i = 1 \ldots \ell$, such that $H^1(X, \mathcal{O}^*_X) = \langle \mathcal{L}_1 \rangle \oplus \cdots \oplus \langle \mathcal{L}_\ell \rangle$, where $\mathcal{L}_i$ is without torsion.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of X and let $h_{\alpha\beta,i} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ be the transition function of $\mathcal{L}_i$. Now, if $S_i \cap U_\alpha = \{f_{\alpha,i} = 0\}$ for some $f_{\alpha,i} \in \mathcal{O}(U_\alpha)$ we get that $f^i_{\alpha\beta} := \frac{f_{\alpha,i}}{f_{\beta,i}}$ are the transition functions of $\mathcal{O}_X(S_i)$. Therefore for each $i = 1, \dots, n$ there exist $\{k^i_j\}_{1\le j\le \ell} \subset \mathbb{Z}$ and $\rho_{\alpha,i} \in \mathcal{O}^*(U_\alpha)$ such that $f^i_{\alpha\beta} = \rho_{\alpha,i}\rho^{-1}_{\beta,i}(h^{k^i_1}_{\alpha\beta,1} \dots h^{k^i_\ell}_{\alpha\beta,\ell})$. Set $\tilde{f}_{\alpha,i} := \frac{f_{\alpha,i}}{\rho_{\alpha,i}}$ and set $\tilde{f}_{\alpha\beta,i} := \frac{\tilde{f}_{\alpha,i}}{\tilde{f}_{\beta,i}}$ we assert that

$$\sum_{1\le i\le n} \lambda_i \frac{d\tilde{f}_{\alpha\beta,i}}{\tilde{f}_{\alpha\beta,i}} = 0.$$

To prove this assertion observe that

$$\sum_{1\le i\le n} \lambda_i \frac{d\tilde{f}_{\alpha\beta,i}}{\tilde{f}_{\alpha\beta,i}} = \sum_{1\le j\le \ell} (\sum_{1\le i\le n} \lambda_i k^i_j) \frac{dh_{\alpha\beta,j}}{h_{\alpha\beta,j}}.$$

We will prove that $\sum_{1\le i\le n} \lambda_i k^i_j = 0$ for every $j = 1, \dots, \ell$. Let $\{\phi_\alpha\}_{\alpha\in\Lambda} \in C^\infty(X)$ be a partition of unity subordinated to the covering $\mathcal{U}$. Then, the first Chern class as a C^∞ 2-form can be given locally by:

$$c_1(\mathcal{L}_i)|_{U_\alpha} = d(\sum_\beta \phi_\beta \frac{dh_{\alpha\beta,i}}{h_{\alpha\beta,i}}) \quad \text{and} \quad c_1(S_i)|_{U_\alpha} = d(\sum_\beta \phi_\beta \frac{d\tilde{f}_{\alpha\beta,i}}{\tilde{f}_{\alpha\beta,i}}). \tag{3.2}$$

Therefore from (3.1) and (3.2) we get that

$$\sum_{1\le j\le \ell} (\sum_{1\le i\le n} \lambda_i k^i_j) c_1(\mathcal{L}_j) = 0.$$

Since $H^1(X, \mathbb{C}) = 0$ we see that the first Chern class, as an application from $H^1(X, \mathcal{O}^*)$ to $H^2(X, \mathbb{Z})$, is injective. Then, $\{c_1(\mathcal{L}_j)\}_j$ are $\mathbb{Z}-$linearly independent in $H^2(X, \mathbb{Z})$, therefore they are $\mathbb{C}-$linearly independent in $H^2(X, \mathbb{C})$ (recall that $H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z}) \otimes \mathbb{C}$). Thus, $\sum_{1\le i\le n} \lambda_i k^i_j = 0$.

Now, we have

$$\sum_{1\le i\le n} \lambda_i \frac{d\tilde{f}_{\alpha\beta,i}}{\tilde{f}_{\alpha\beta,i}} = 0$$

and therefore we can define a global meromorphic 1-form with simple poles at S by:

$$\eta|_{\mathcal{U}_\alpha} = \sum_{1\le i\le n} \lambda_i \frac{d\tilde{f}_{\alpha,i}}{\tilde{f}_{\alpha,i}}.$$

Then, $\Omega - \eta$ is a global holomorphic 1-form in X. But $H^1(X, \mathbb{C}) = 0$ implies that there is not non-trivial global holomorphic 1-form (recall that, by the Hodge decomposition theorem, $H^1(X, \mathbb{C}) = H^1(X, \mathcal{O}_X) \oplus H^0(X, T^*X)$), therefore $\Omega \equiv \eta$ and this implies that $\mathcal{F}$ is logarithmic. □

Let X be a compact algebraic surface, let $\mathcal{F}$ be a holomorphic foliation and let S be an invariant compact curve. In the next two lemmas we give sufficient conditions to ensure that $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S)$ is holomorphically trivial.

Lemma 3.2. *Let X; $\mathcal{F}$ and S be as above. Suppose that the Picard group of X is torsion free and $H^1(X, \mathbb{C}) = 0$. In addition suppose*

i) $S^2 > 0$.

ii) For each $p \in \mathrm{Sing}(\mathcal{F}) \cap S$ any local separatrix of $\mathcal{F}$ through p is a local branch of S at p.

iii) The singularities of $\mathcal{F}$ in S are generalized curves.

iv) $\sum_{p \in \mathrm{Sing}(\mathcal{F})-S} BB_p(\mathcal{F}) \geq 0$.

Then, $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S)$ is holomorphically trivial.

Proof. Since every singularity of $\mathcal{F}$ in S is a non-dicritical generalized curve and since every local separatrix of $\mathcal{F}$ through any $p \in \mathrm{Sing}(\mathcal{F}) \cap S$ is a local branch of S we have $GSV(\mathcal{F}, S) = 0$ (see [Br2]). Then, by condition $i\_$ and proposition 2.1, we obtain that $BB(\mathcal{F}) \leq S^2$.

On the other hand, if $CS_p(\mathcal{F}, S)$ is the Camacho-Sad index associated to the singularity $p \in S$ we have $CS_p(\mathcal{F}, S) = BB_p(\mathcal{F})$ (see [Br2]). Thus, by condition $iv\_$ and Camacho-Sad index theorem ([CS]), we conclude that $S^2 \leq BB(\mathcal{F})$. Hence $c_1^2(N_{\mathcal{F}}) = BB(\mathcal{F}) = S^2$.

Now, by (2.3), we obtain $c_1(N_{\mathcal{F}})S = S^2$. Therefore, we have the equalities

$$c_1^2(N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S)) = 0 \quad \text{and} \quad c_1(N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S))S = 0.$$

Then, by Hodge Index Theorem we get that $c_1(N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S))$ is homologous to 0 in $H^2(X, \mathbb{Q})$. As consequence, for some $m \in \mathbb{N}$, $mc_1(N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S))$ is homologous to 0 in $H^2(X, \mathbb{Z})$. On the other hand, since $H^1(X, \mathbb{C}) = 0$ we obtain that $H^1(X, \mathcal{O}_X) = 0$ by the Hodge decomposition theorem. Therefore, the first Chern class as an application from $H^1(X, \mathcal{O}^*)$ to $H^2(X, \mathbb{Z})$ is injective. Thus $m(N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S))$ is a holomorphically trivial line bundle and this implies that $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S)$ is a holomorphically trivial line bundle since $H^1(X, \mathcal{O}^*)$ is a torsion free group. □

Remark 3.1. For the proof of the following lemma we recall that, by a theorem of Kodaira, a compact complex surface is algebraic if, and only if, there exists a holomorphic line bundle $\mathcal{M}$ with $c_1^2(\mathcal{M}) > 0$ (see [BPV] page 126). Therefore, if X is a compact algebraic surface with Picard group generated by the line bundle $\mathcal{L}$ we have $c_1^2(\mathcal{L}) > 0$.

Lemma 3.3. *Let X, $\mathcal{F}$ and S as the former lemma. Suppose that the Picard group of X is isomorphic to $\mathbb{Z}$, $GSV(\mathcal{F}, S) \geq 0$ and $BB(\mathcal{F}) \leq S^2$. Then, $N_{\mathcal{F}}^* \otimes \mathcal{O}_X(S)$ is holomorphically trivial.*

Proof. By Baum-Bott formula (2.2) we know that $c_1^2(N_{\mathcal{F}}) = BB(\mathcal{F})$ and by (2.3) $c_1(N_{\mathcal{F}})S \geq S^2$. Then

$$\begin{aligned} c_1^2(N_{\mathcal{F}}^* \otimes \mathcal{O}_X(S)) &= c_1^2(N_{\mathcal{F}}) + S^2 - 2c_1(N_{\mathcal{F}})S \\ &\leq c_1^2(N_{\mathcal{F}}) - S^2 \leq 0 \end{aligned}$$

Hence, $c_1^2(N_{\mathcal{F}}^* \otimes \mathcal{O}_X(S)) \leq 0$. But then, we have $N_{\mathcal{F}}^* \otimes \mathcal{O}_X(S) = k\mathcal{L}$, where $k \in \mathbb{Z}$ and $\mathcal{L}$ is a line bundle which generates $Pic(X)$ as a group.

Now,

$$k^2 c_1^2(\mathcal{L}) = c_1^2(N_{\mathcal{F}}^* \otimes \mathcal{O}_X(S)) \leq 0.$$

Then, $k = 0$. Therefore $N_{\mathcal{F}}^* \otimes \mathcal{O}_X(S)$ must be a holomorphically trivial line bundle. □

Before proving theorem A let us see a simple extension of theorem 1 in [CL] to compact algebraic surfaces X with Picard group isomorphic to $\mathbb{Z}$ and $H^1(X, \mathbb{C}) = 0$.

Proposition 3.1. *Let $\mathcal{F}$ be a holomorphic foliation on a compact algebraic surface X with $H^1(X, \mathbb{C}) = 0$ and $Pic(X) = \mathbb{Z}$. Let S be an invariant compact curve with only nodal type singularities. If $BB(\mathcal{F}) \leq S^2$, then $\mathcal{F}$ is logarithmic.*

Proof. Note that $GSV(\mathcal{F}, S) \geq 0$ since S has only nodal type singularities (this is consequence of, for example, the algebraic index formula given in [G]). Therefore lemma 3.3 together with lemma 3.1 give the proposition. □

Recall that a singularity is said non-degenerated if the linear part of the vector field which locally induces the foliation has non-zero eigenvalues.

Corollary 3.1. *Let $\mathcal{F}$ be a holomorphic foliation on a compact algebraic surface X with $H^1(X, \mathbb{C}) = 0$ and $Pic(X) = \mathbb{Z}$. Let S be an invariant compact curve with only nodal type singularities. If $\mathrm{Sing}(\mathcal{F}) \cap S = \mathrm{Sing}(S)$ and the singularities of $\mathcal{F}$ in S are non-degenerated, then $\mathcal{F}$ is logarithmic.*

Proof. By using the algebraic index formula given in [G] we get $GSV(\mathcal{F}, S) = 0$. By remark 3.1 we see that $S^2 > 0$. Therefore by proposition 2.1, $BB(\mathcal{F}) \leq S^2$. Now, by the former proposition we conclude that $\mathcal{F}$ is logarithmic. □

The next lemma will be used in the proof of theorem A. First let us introduce some notation.

Let X be a compact algebraic surface and let S be a compact curve on X. Set $S_0 = S$, $X_0 = X$ and let S_{i+1} be the strict transformed curve of S_i by π_{i+1}, where π_{i+1} is the blow up of the surface X_i with center $p_i \in S_i$.

Let $D_{i,0}$ be the exceptional line introduced in X_i by π_i. For each $j \geq i+1$ let $D_{i,j-i}$ be the strict transform of $D_{i,0}$ by the composition $\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_j : X_j \to X_i$. Let $\delta_{i,k}$ be the algebraic multiplicity of $D_{i,k}$ in p_{i+k}. Then, by elementary properties of the blow up, $\pi^*_{i+k+1}(D_{i,k}) = D_{i,k+1} + \delta_{i,k} D_{i+k+1,0}$. In fact, $\pi^*_{i+k+1}(D_{i,k}) = D + \delta_{i,k} D_{i+k+1,0}$, where D is the strict transform of $D_{i,k}$ by π_{i+k+1} (see [GH] page 475). On the other hand, since $D_{i,k}$ is the strict transform of $D_{i,0}$ by $\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_{i+k}$ we get that D is the strict transform of $D_{i,0}$ by $\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_{i+k} \circ \pi_{i+k+1}$. That is $D = D_{i,k+1}$.

Let us denote $D_k := \cup_{i+j=k} D_{i,j}$.

Lemma 3.4. *Let X, S and D_n be as above. Let $\mathcal{F}$ be a holomorphic foliation on X with a singularity in $p_0 \in S$ which is a generalized curve. In addition suppose that any local separatrix of $\mathcal{F}$ through p_0 is a local branch of S. Then,*

$$N^*_{\mathcal{F}_n} \otimes \mathcal{O}_{X_n}(S_n \cup D_n) = (\pi_1 \circ \cdots \circ \pi_n)^*(N^*_{\mathcal{F}} \otimes \mathcal{O}_X(S)),$$

where $\mathcal{F}_n$ is the strict transformed foliation of $\mathcal{F}$ and S_n is the strict transformed curve of S by $\pi_1 \circ \cdots \circ \pi_n$.

Proof. We use the notation introduced in the above paragraph.

Set $\mathcal{F}_0 := \mathcal{F}$ and let $\mathcal{F}_{i+1}$ be the strict transformed foliation of $\mathcal{F}_i$ by π_{i+1}, where $i = 0, ..., n-1$. We know that $\pi_i^*(S_{i-1}) = S_i + \mu_{i-1} D_{i,0}$, where μ_{i-1} is the algebraic multiplicity of S_{i-1} at p_{i-1} (see [GH] page 475).

Since every local separatrix of $\mathcal{F}_i$ through any $p_i \in \mathrm{Sing}(\mathcal{F}_i) \cap (S_i \cup D_i)$ is a local branch of $S_i \cup D_i$ and p_i is a generalized curve, we have by [CLS] theorem 3:

$$m_{p_i} + 1 = \mu_i + \sum_{\substack{k+j=i,\\ k \geq 1}} \delta_{k,j} \quad \text{for every } i,$$

where m_{p_i} is the algebraic multiplicity of $\mathcal{F}_i$ at p_i and $\delta_{k,j}$ is the algebraic multiplicity of $D_{k,j}$ in p_{j+k}.

Now, we can prove the lemma by induction in the number of blow ups: If $n = 0$ there is nothing to prove. Let us suppose that the lemma is true for every $k < n$ and we will prove that it is also true for n.

$$\begin{aligned}
N^*_{\mathcal{F}_n} \otimes \mathcal{O}(S_n) \otimes \mathcal{O}(\cup_{i+j=n} D_{i,j}) &= \pi_n^*(N^*_{\mathcal{F}_{n-1}} \otimes \mathcal{O}(S_{n-1})) \otimes \\
&\quad \otimes (m_{p_{n-1}} - \mu_{n-1} + 1) D_{n,0} \otimes \mathcal{O}(\cup_{i+j=n, j\geq 1} D_{i,j}) \\
&= \pi_n^*(N^*_{\mathcal{F}_{n-1}} \otimes \mathcal{O}(S_{n-1})) \otimes (m_{p_{n-1}} - \mu_{n-1} + 1) D_{n,0} \otimes \\
&\quad \otimes \mathcal{O}(\cup_{i+j=n, j\geq 1} \{\pi_n^*(D_{i,j-1}) - \delta_{i,j-1} D_{n,0}\}) \\
&= \pi_n^*(N^*_{\mathcal{F}_{n-1}} \otimes \mathcal{O}(S_{n-1}) \otimes \mathcal{O}(\cup_{i+j=n, j\geq 1} D_{i,j-1})) \otimes \\
&\quad \otimes (m_{p_{n-1}} - \mu_{n-1} + 1 - \sum_{i+j=n, j\geq 1} \delta_{i,j-1}) D_{n,0} \\
&= \pi_n^*(N^*_{\mathcal{F}_{n-1}} \otimes \mathcal{O}(S_{n-1}) \otimes \mathcal{O}(\cup_{i+j=n-1} D_{i,j})) \\
&= \pi_n^* \circ \pi_{n-1}^* \circ \cdots \circ \pi_1^*(N^*_{\mathcal{F}_0} \otimes \mathcal{O}(S_0)) \qquad \square
\end{aligned}$$

Now, we can prove theorem A.

Proof of theorem A. It is well known that there exists a sequence of blow ups at points of S

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \to X_1 \xrightarrow{\pi_1} X$$

such that $\tilde{S} \cup D$ is a normal crossing divisor on X_n, where $\tilde{S}$ is the strict transform of S and D is the exceptional divisor introduced by the above sequence of blow ups. Then, by the former lemma $N^*_{\tilde{\mathcal{F}}} \otimes \mathcal{O}(\tilde{S} \cup D) = (\pi_1 \circ \cdots \circ \pi_n)^*(N^*_{\mathcal{F}} \otimes \mathcal{O}(S))$.

Observe that $GSV(\mathcal{F}, S) = 0$ and $S^2 > 0$ by [Br2] and remark 3.1, respectively. Therefore, by proposition 2.1 we get that $BB(\mathcal{F}) \leq S^2$.

Now, by lemma 3.3 if we are in case $i\_$, or by lemma 3.2 if we are in case $ii\_$, we get that $N^*_{\mathcal{F}} \otimes \mathcal{O}(S)$ is holomorphically trivial. Thus, $N^*_{\tilde{\mathcal{F}}} \otimes \mathcal{O}(\tilde{S} \cup D)$ is holomorphically trivial. Finally, by lemma 3.1, we get that $\tilde{\mathcal{F}}$ is logarithmic and this implies that $\mathcal{F}$ is logarithmic. $\square$

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